MATH 181E: Mathematical Statistics - Time Series
Discussion 2:

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Outline

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Review

Exercise Discussion

Definition

Suppose $\{X_t\}$ is a time series with $E(X_t^2) < \infty$

- Mean function is defined as

$$\mu_X(t) = E(X_t)$$

- Autocovariance function (ACVF) is defined as

$$\gamma_X(s,t) = \mathrm{Cov}\left(X_s, X_t\right) = \mathrm{E}\left[\{X_s - \mu_X(s)\}\{X_t - \mu_X(t)\}\right]$$

- Autocorrelation function (ACF) is defined as

$$\rho_X(s,t) = \frac{\gamma_X(s,t)}{\sqrt{\gamma_X(s,s)\gamma_X(t,t)}}$$

Definition

In this class, we claim $\{X_t\}$ is stationary which means weakly stationary time series and satisfies

- $\mu_X(t)$ is independent of t
- $\gamma_X(t+h, t)$ is independent of t for each h

And we can simply write $\mu_X = \mu_X(t)$.

- Autocovariance function at lag h is

$$\gamma_X(h) = \gamma_X(h, 0) = \operatorname{Cov}(X_{t+h}, X_t)$$

- Autocorrelation function at lag h is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \operatorname{Corr}(X_{t+h}, X_t)$$

Properties

Properties of $\gamma_X(\cdot)$:

1. $\gamma_X(0) \geq 0$
2. $|\gamma_X(h)| \leq \gamma(0)$ for all h
3. $\gamma_X(h) = \gamma_X(-h)$ for all h
4. γ_X is nonnegative definite; i.e., $\sum_{i,j=1}^{n} a_i \gamma_X(i-j) a_j \geq 0$ for all positive integers n and real vectors $\boldsymbol{a} = (a_1, \ldots, a_n)^{\mathrm{T}} \in \mathbb{R}^n$.

Properties of $\rho_X(\cdot)$:

1. all properties of $\gamma_X(\cdot)$
2. $\rho_X(0) = 1$

Outline

Exercise 1

Consider the random walk with drift model

$$x_t = \delta + x_{t-1} + w_t,$$

for $t = 1, 2, \ldots$, with $x_0 = 0$, where w_t is white noise with variance σ_w^2.

1. Show that the model can be written as $x_t = \delta t + \sum_{k=1}^{t} w_k$.
2. Find the mean function and the autocovariance function of x_t.
3. Argue that x_t is not stationary.
4. Discuss $y_t = x_t - x_{t-1}$ and check whether the new time series is stationary.

Exercise 2

A time series with a periodic component can be constructed from

$$x_t = U \sin\left(2\pi\omega_0 t\right)$$

where U are independent random variables with zero means and $\mathrm{E}\left(U^2\right) = \sigma^2$. Check whether this series is weakly stationary and calculate the autocovariance function $\gamma(h)$.