MATH 181E: Mathematical Statistics - Time Series
Discussion 3:

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Outline

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Asymptotic Normal Distribution

Theorem A.5: For the linear process

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j \psi_j W_{t-j}$$

with white noise $\{W_t\} \sim \mathrm{IID}\left(0, \sigma^2\right)$ and $\sum_{j=-\infty}^{\infty} \psi_j \neq 0$, we have

$$\sqrt{n}\left(\bar{X}_n - \mu_X\right) \sim \mathrm{AN}(0, \nu),$$

where

$$\nu = \sum_{j=-\infty}^{\infty} \gamma_X(h) = \sigma^2 \left(\sum_{j=-\infty}^{\infty} \psi_j\right)^2 < \infty.$$

Asymptotic Normal Distribution

Theorem A.7: For the linear process

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j},$$

with $\{W_t\} \sim \mathrm{IID}\left(0, \sigma^2\right), \sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and $\mathrm{E}\left(W_t^4\right) < \infty$, we have, for fixed h,

$$\hat{\rho}_X(h) = \begin{pmatrix} \hat{\rho}_X(1) \\ \vdots \\ \hat{\rho}_X(h) \end{pmatrix} \sim \mathrm{AN}\left\{ \rho_X(h) = \begin{pmatrix} \rho_X(1) \\ \vdots \\ \rho_X(h) \end{pmatrix}, n^{-1}\Omega \right\}$$

where $\mathbf{\Omega} = [\omega_{ij}]_{i,j=1}^{h}$ is the covariance matrix whose (i,j)-element is given by Bartlett's formula,

$$\omega_{ij} = \sum_{k=1}^{\infty} \{\rho_X(k+i) + \rho_X(k-i) - 2\rho_X(i)\rho_X(k)\} \times \\ \{\rho_X(k+j) + \rho_X(k-j) - 2\rho_X(j)\rho_X(k)\}$$

Remark

- If $\rho_X(i) = 0$ for all $i \neq 0$, then $\boldsymbol{\Omega} = \boldsymbol{I}_h$.
- For each i, it holds that

$$\hat{\rho}_X(i) \sim \mathrm{AN}\left(\rho_X(i), n^{-1}\omega_{ii}\right)$$

or equivalently

$$\sqrt{n}\left[\hat{\rho}_X(i) - \rho_X(i)\right] \sim \mathrm{AN}\left(0, \omega_{ii}\right)$$

where

$$\omega_{ii} = \sum_{k=1}^{\infty} \left\{\rho_X(k+i) + \rho_X(k-i) - 2\rho_X(i)\rho_X(k)\right\}^2$$

Confidence Interval

Let X be a random sample from a probability distribution with statistical parameter θ. A confidence interval for the parameter θ, with confidence level $1-\alpha$, is an interval $(u(X), v(X))$ determined by random variables $u(X)$ and $v(X)$ with the property:

$$P\{u(X) < \theta < v(X)\} = \alpha.$$

But in our course, we may discuss the approximate confidence interval. We can use the asymptotic distribution to estimate the true distribution of $u(X)$ and $v(X)$. For example,

$$\sqrt{n}\left[\hat{\rho}_X(i) - \rho_X(i)\right] / \sqrt{w_{ii}} \sim \mathrm{AN}\left(0, 1\right)$$

has a $100(1-\alpha)\%$ confidence interval

$$\frac{n\left[\hat{\rho}_X(i) - \rho_X(i)\right]^2}{w_{ii}^2} \leq z_{\alpha/2}^2$$