MATH 181E: Mathematical Statistics - Time Series
Discussion 4:

Dehao Dai

Department of Mathematics, UCSD

February 2, 2023

Outline

Review

Exercise

Determine the p and q in ARMA

ARMA(p, q) process $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q}$$

where $\{W_t\} \sim WN\left(0, \sigma^2\right)$. It can also be written as

$$\phi(B) X_t = \theta(B) W_t$$

where

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

and

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q.$$

Causality and Invertibility

Theorem

A unique stationary solution to $\phi(B)X_t = \theta(B)W_t$ *exists if and only if the roots of* $\phi(z)$ *avoid the unit circle:*

$$|z| = 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0$$

Theorem

An ARMA(p,q) *process* $\{X_t\}$ *is causal if and only if* $\phi(z) \neq 0$ *for all* $|z| \leq 1$.

Theorem

An ARMA(p,q) *process* $\{X_t\}$ *is invertible if and only if* $\theta(z) \neq 0$ *for all* $|z| \leq 1$

Causal function of ARMA(2,1)

We have $\phi(z)$ with roots z_1 and z_2. If $|z_1|, |z_2| > 1$, then $\{X_t\}$ is causal. To find the causal function, we have

$$\begin{aligned}
&1 = \psi_0, \\
&\theta_1 = \psi_1 - \phi_1\psi_0 \\
&\theta_2 = \psi_2 - \phi_1\psi_1 - \phi_2\psi_0
\end{aligned}$$

We have the solution:

$$\psi_0 = 1, \quad \psi_1 = \theta_1 + \phi_1\psi_0$$

and

$$\psi_j - \phi_1\psi_{j-1} - \phi_2\psi_{j-2} = 0, \quad j \geq 2$$

Causal function of ARMA(2,1)

We have several cases:

1. z_1 and z_2 are different real numbers
2. z_1 and z_2 are conjugate complex numbers
3. z_1 and z_2 are same real numbers

The corresponding general solutions are

1. $\psi_n = c_1 z_1^{-n} + c_2 z_2^{-n}$
2. If $z_1 = |z_1|(\cos\theta + i\sin\theta)$, then
$$\psi_n = c_1|z_1|^{-n}\cos(n\theta - c_2)$$
3. $\psi_n = (c_1 + c_2 n) z_1^{-n}$

for $n \geq \max(p = 2, q + 1 = 1 + 1) - p = 0$.
The constants c_1 and c_2 are found from the boundary conditions $\psi_0 = 1$ and ψ_1.

Outline

Review

Exercise

Exercise 1

Determine which of the following processes are ARMA(p,q) and find the p and q. And determine which of them are causal and/or invertible:

(a) $X_t + .2X_{t-1} - .48X_{t-2} = W_t$,

(b) $X_t + 1.9X_{t-1} + .88X_{t-2} = W_t + .2W_{t-1} + .7W_{t-2}$,

(c) $X_t + .6X_{t-2} = W_t + 1.2W_{t-1}$,

(d) $X_t + 1.8X_{t-1} + .81X_{t-2} = W_t$,

(e) $X_t + 1.6X_{t-1} = W_t - .4W_{t-1} + .04W_{t-2}$.

Exercise 2

Find the coefficients $\psi_j, j = 0, 1, 2, \ldots$, in the representation

$$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}$$

of the ARMA(2,1) process,

$$\left(1 - .5B + .04B^2\right) X_t = (1 + .25B) W_t, \quad \{W_t\} \sim \mathrm{WN}\left(0, \sigma^2\right)$$