MATH 181E: Mathematical Statistics - Time Series
Discussion 6:

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February 17, 2023

Outline

Forecast ARMA based on Infinite past

WLOG, $\mu_X = 0$, ARMA is causal and invertible. Thus write

$$X_{n+m} = \sum_{j=1}^{\infty} \psi_j W_{n+m-j} + W_{n+m}, \quad W_{n+m} = \sum_{j=1}^{\infty} \pi_j X_{n+m-j} + X_{n+m}$$

Then

$$\begin{aligned} \tilde{X}_{n+m} &= \mathrm{E}\left(X_{n+m} \mid X_n, X_{n-1}, \ldots\right) \\ &= -\sum_{j=1}^{m-1} \pi_j \tilde{X}_{n+m-j} - \sum_{j=m}^{\infty} \pi_j X_{n+m-j} \end{aligned}$$

If $m = 1$, we can specify this as

$$\tilde{X}_{n+1} = -\sum_{j=1}^{\infty} \pi_j X_{n+m-j} (= X_{n+1} - W_{n+1})$$

Mean square error for infinite-past prediction

The MSE for prediction is

$$\mathrm{E}\left[\left(X_{n+m} - \tilde{X}_{n+m}\right)^2\right] = \mathrm{E}\left(\sum_{j=0}^{m-1} \psi_j W_{n+m-j}\right)^2 = \sigma^2 \sum_{j=0}^{m-1} \psi_j^2$$

Specifically, if $m = 1$, then $\mathrm{E}\left[\left(X_{n+m} - \tilde{X}_{n+m}\right)^2\right] = \sigma_W^2$

Truncated m-step-ahead forecast

For ARMA(p,q) models, the truncated predictors are

(i)

$$\tilde{x}^n_{n+m} = \phi_1 \tilde{x}^n_{n+m-1} + \cdots + \phi_p \tilde{x}^n_{n+m-p} + \theta_1 \tilde{w}^n_{n+m-1} + \cdots + \theta_q \tilde{w}^n_{n+m-q},$$

(ii) $\tilde{x}^n_t = x_t$ for $1 \le t \le n$

(iii) $\tilde{x}^n_t = 0$ for $t \le 0$.

The truncated prediction errors are given by:

(i) $\tilde{w}^n_t = 0$ for $t \le 0$ or $t > n$

(ii)

$$\tilde{w}^n_t = \phi(B)\tilde{x}^n_t - \theta_1 \tilde{w}^n_{t-1} - \cdots - \theta_q \tilde{w}^n_{t-q}$$

for $1 \le t \le n$.