MATH 181E: Mathematical Statistics - Time Series
Discussion 8: Review of Midterm Exam 2

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Outline

Invertible function of ARMA

ARMA(p,q) has the form $\phi(B)X_t = \theta(B)W_t$, where

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

and

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q.$$

Suppose that this process is invertible, we can get the invertible function, i.e. the coefficients of X_t. First, we can express W_t w.r.t X_t, given by

$$W_t = \frac{\phi(B)}{\theta(B)}X_t = \sum_{j=0}(c_1 z_1^{-j} + c_2 z_2^{-j} + \cdots + c_q z_q^{-j})\phi(B)X_t$$

where z_i is the root of $\theta(z) = 0$ and the summation is from

$$\frac{1}{(1 - z/z_1)(1 - z/z_2)\cdots(1 - z/z_q)} = \sum_{j=0}^{q}(z_1)^{-j}\sum_{j=0}^{q}(z_2)^{-j}\cdots\sum_{j=0}^{q}(z_q)^{-j}z^j$$

Invertible function of ARMA

π_j could be found from $W_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$. Then we can define the m-step truncated prediction $\tilde{X}_{n+1}^n = -\sum_{i=1}^n \pi_j X_{n+1-j}$.

Remark

1. We need to tell the notation from $\tilde{X}_{n+1}$, which means the m-step prediction based on the infinite past.
2. Backcast prediction is similar to the forecast prediction. In general, you can use the method of moment to check whether the formula is BLP.
3. Sometimes, you do not need to figure out what π_j exactly is. You can refer to notes from Lecture 19. But you need to get the two recursive equation from $\tilde{W}_t$ and $\tilde{X}_t$.

ARIMA(p,d,q)

A process x_t is said to be ARIMA(p, d, q) if

$$\nabla^d x_t = (1 - B)^d x_t$$

is $\mathrm{ARMA}(p, q)$. In general, we will write the model as

$$\phi(B)(1 - B)^d x_t = \theta(B) w_t.$$

If $\mathrm{E}\left(\nabla^d x_t\right) = \mu$, we write the model as

$$\phi(B)(1 - B)^d x_t = \delta + \theta(B) w_t,$$

where $\delta = \mu\,(1 - \phi_1 - \cdots - \phi_p)$.

ARIMA(p,d,q)

The mean-squared prediction error can be approximated by

$$P^n_{n+m} = \sigma^2_w \sum_{j=0}^{m-1} \psi^{*2}_j$$

where ψ^*_j is the coefficient of z^j in

$$\psi^*(z) = \theta(z)/\phi(z)(1-z)^d.$$

If $p = d = 1$,$q = 0$, then we can simply the $\psi^*(z)$ as

$$\psi^*_0 = 1, \quad \psi^*_1 = 1$$

and

$$\psi^*_j - (1+\phi)\psi^*_{j-1} + \phi\psi^*_{j-2} = 0 \text{ for } j \geq 2$$

ARIMA(p,d,q)

Recall the method we used in causal function. We can introduce the characteristic function

$$x^2 - (1+\phi)x + \phi = 0$$

Now you check the two different roots of the equation above. And you can imagine the relationship between the roots and the causal functions or the ψ_j^*.